Renormalization of gauge theory

connected graphs, 1-particle irreducible (proper) graphs

- generating functional of n-point graphs

$$Z[J] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\left(S + \int d^4x \, J(x)\phi(x)\right)}$$

n-point graph : $\displaystyle \left.\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}$

- generating functional of n-point connected graphs

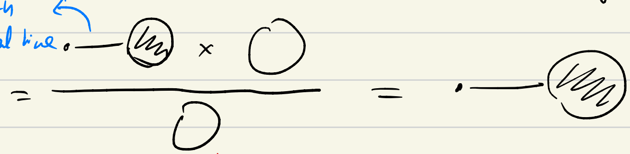
$$W[J] = -i\hbar \ln Z[J] \quad , \quad \text{or} \quad Z[J] = e^{\frac{i}{\hbar}W[J]}$$

n-point connected graph : $\displaystyle \left.\frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}$

Ex : 1-point connected graph

$$\left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} = \frac{\hbar}{i} \frac{\left.\frac{\delta Z}{\delta J(x)}\right|_{J=0}}{Z[J]|_{J=0}}$$

connected graph with 1 external line



all graphs without external line

- generating functional of proper graphs

$\displaystyle \phi_0(x) = \frac{\delta W}{\delta J(x)}$, "classical fields", unfortunate name, because there is quantum correction

not taking $J(x)$ to 0

from $\frac{\delta W[J]}{\delta J(x)} = \phi_0(x)$, one obtains $J = J[\phi]$ in principle

and then $\underline{\Gamma[\phi_0]} = W[J[\phi_0]] - \int d^4x \, J\phi_0$

$\downarrow$ affective action, generating functional of proper graphs

n-point proper graph: $\frac{\delta^n \Gamma}{\delta\phi_0(x_1) \cdots \delta\phi_0(x_n)}$

note: 1 $\frac{\delta \Gamma}{\delta J(x)} = \frac{\delta W[J]}{\delta J(x)} - \phi_0(x) = 0$ by definition

Γ deosn't depend on $J(x)$, which is similar to Legendre transformation in classical mechanics

2. $\frac{\delta \Gamma}{\delta \phi_0(x)} = -J(x)$, similar to classical equation of motion

3. at tree level Γ_{tree} is the classical action

Ward identity

$$Z[J] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^4x (\mathcal{L}[\phi] + J\phi)}$$

assume $\mathcal{D}\phi$ and $\mathcal{L}[\phi]$ inv. under $\phi \to \phi'$

$\phi = \phi' - \varepsilon g(\phi')$

$$Z[J] = \int \mathcal{D}\phi' \, e^{\frac{i}{\hbar}\int d^4x (\mathcal{L}[\phi'] + J\phi)}$$

inv. of $\mathcal{D}\phi$, $\mathcal{L}[\phi]$

$$\therefore \; = \int \mathcal{D}\phi' \, e^{\frac{i}{\hbar}\int d^4x [\mathcal{L}[\phi'] + J(\phi' - \varepsilon g(\phi'))]}$$

Shakespeare thm, ϕ' is a dummy var.

$$= \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^4x \left(\mathcal{L}[\phi] + J\phi - \varepsilon J g(\phi) \right)}$$

$$\Rightarrow \int \mathcal{D}\phi \int d^4y \left(J g(\phi) \right) e^{\frac{i}{\hbar}\int d^4x \left(\mathcal{L}[\phi] + J\phi \right)} = 0$$

or $$\left\langle \int d^4y \, J(y) \, g(\phi(y)) \right\rangle_J = 0$$

multiplicative perturbative renormalization in general

perturbative:

given unrenormalized proper Green function, construct corresponding finite renormalized proper Green function loop by loop.

- assume all $(n-1)$-loop ones are made finite
- determine all divergences in n-loop proper graphs

multiplicative

- n-loop divergences can be absorbed by rescaling $(n-1)$-loop renormalized fields and parameters

differences between non-Abelian gauge theory and $\lambda\phi^4$

in $\lambda\phi^4$ theory, 2, 3, 4-point functions can have different Z factors (ϕ^2, ϕ^3, ϕ^4 in $\mathcal{L}$ are independent)

in YM, Z factors of 2, 3, 4-point functions are related ($F_{\mu\nu}^2$ contains A^2, A^3, A^3), constraint by BRST, less Z-factors.

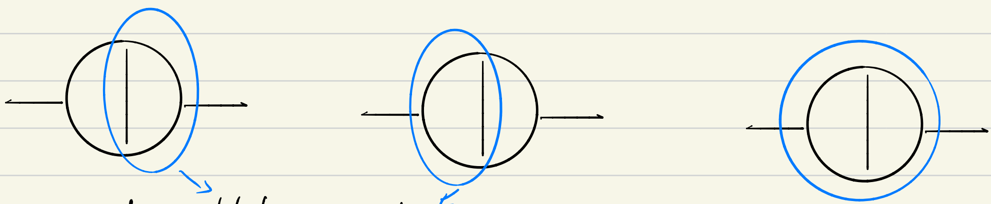
fact:

1 in YM, all divergences at n-loop are local (spacetime integrals of local polynomials in fields and derivatives)

2. power counting: 2, 3, 4 proper graphs can be divergent.

Topology of graphs

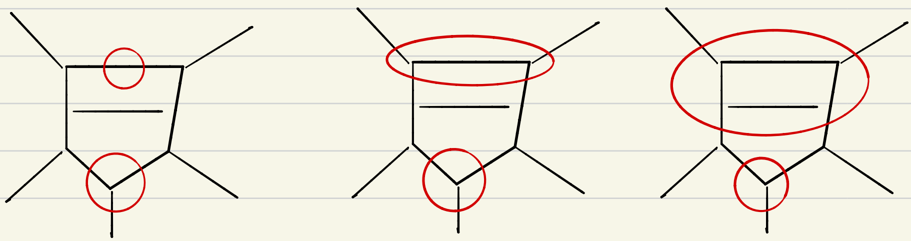
any connected graph



draw blobs around all 2, 3, 4 proper subgraphs (potentially divergent)

maximal potentially divergent proper graph

uniqueness: identifying the set of maximal potentially div. proper graphs in a connected graph

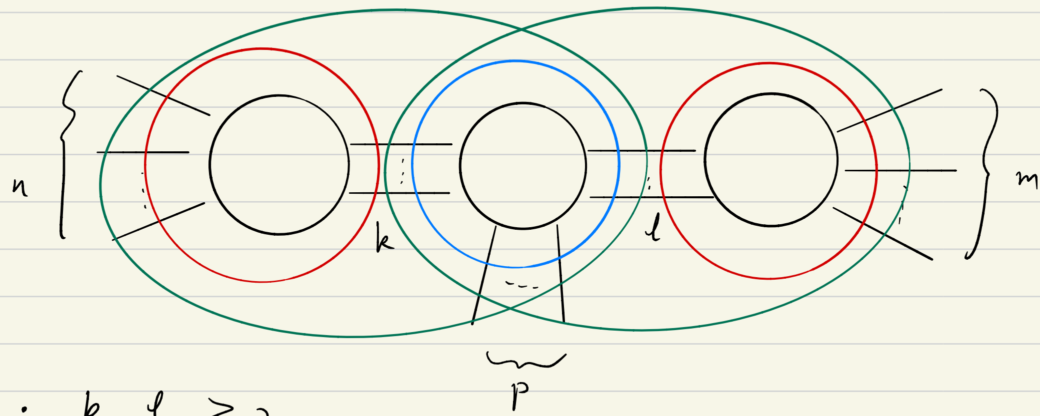


Thm: blobs around maximal potentially divergent proper subgraphs are unique and do not intersect

proof: assume the contrary

- then there are at least 2 blobs which are overlapping, each is either 2, 3, 4-point and maximal (green)

- draw a blob around intersecting vertices (blue), two blobs around remaining parts (red)



- $k, l \geq 2$

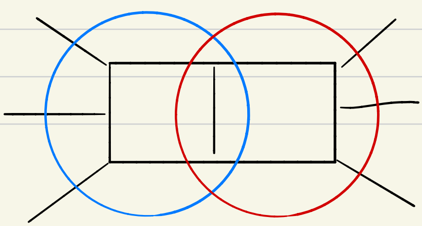
$$k + p + m \leq 4 \quad , \quad l + p + n \leq 4$$

$$\Rightarrow \quad p + m \leq 2 \qquad p + n \leq 2$$

$$\Rightarrow \quad 2p + m + n \leq 4 \quad \Rightarrow \quad m + n + p \leq 4$$

- the union of two green blobs is again potentially div. proper graph ($\# \leq 4$)
- uniqueness follows from non-overlapping

note: not work for 5-point and above, but they are not power counting div in 4 dim



Thm: Weinberg

if all subgraphs of a connected graph are finite (by power counting / renormalization), the graph has only local overall divergences in 2, 3, 4-point

renormalization of proper graphs $\Rightarrow$ renorm. of connected graphs

Ward identities

$$\mathcal{L}_{qn} = -\frac{1}{4}(F_{\mu\nu}^{a})^2 - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 - (\partial^\mu b_a)(D_\mu c)^a$$

- Lorentz gauge, inv. under global part of the gauge symm.
- renormalizability only proved explicitly for Lorentz inv. rigid symmetry inv. gauges

Assuming finiteness of the effective action up to $(n-1)$-loop rescale fields:

$$A_\mu^a = \sqrt{Z_3}\, A_\mu^{a,\,ren} \qquad b_a = \sqrt{Z_{gh}}\, b_a^{ren} \qquad c^a = \sqrt{Z_{gh}}\, c^{a,\,ren}$$

$$g = \frac{Z_1}{(Z_3)^{3/2}}\mu^{\frac{1}{2}(4-n)} u \quad , \quad \xi = Z_\xi\, \xi^{ren}$$

- in action, b_a c^a appear in pairs, only $\sqrt{Z_b}\sqrt{Z_c}$ is meaningful, So we set $Z_b = Z_c \equiv Z_{gh}$
- one needs Z_ξ even ξ can be put to 1.

  Ex: 1-loop proper self-energy graph

$$\Rightarrow \quad \langle A_\mu^a A_\nu^b \rangle = \underbrace{(\eta_{\mu\nu}k^2 - k_\mu k_\nu)}_{\text{transversal piece}}\delta^{ab}\,\Pi(k^2)$$

$\Rightarrow$ renormalization of $-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2$

$\Rightarrow$ 1-loop, no renorm. for $-\frac{1}{2\xi}(\partial^\mu A_\mu^a)^2$

however, $A_\mu^a \longrightarrow Z_3 A_\mu^{a,\,ren}$, needs $\xi \rightarrow \sqrt{Z_\xi}\,\xi$

so overall the gauge fixing term is unrenorm.

and $Z_3 = Z_\xi$ at 1-loop

- actually one can prove $\langle A_\mu^a A_\nu^b \rangle \sim (\eta_{\mu\nu} k^2 - k_\mu k_\nu)\Pi(k^2)$ up to any loop by using a Ward identity
- $\mathcal{L}_{fix}$ is not renorm. at any loop : $Z_\xi = Z_3$

extra term

BRST transf rules are non-linear in fields (diff. from QED or linear σ-model)

$$\delta_B A_\mu^a = \partial_\mu c^a \Lambda + g f_{bc}{}^a A_\mu^b c^c \Lambda$$

but $\langle g f_{bc}{}^a A_\mu^b c^c \rangle \neq g f_{bc}{}^a \langle A_\mu^b \rangle \langle c^c \rangle$

to derive Ward identity, we shall encounter $\langle \delta_B A_\mu^a \rangle$

trick : add extra source term for $\delta_B A_\mu^a$, $\delta_B c^a$

$$\mathcal{L}_{extra} = K_a^\mu \delta_B A_\mu^a + L_a \delta_B c^a = \underbrace{K_a^\mu}_{\text{anti-commuting}} D_\mu c^a + \underbrace{L_a}_{\text{commuting}} \tfrac{1}{2} g f_{bc}{}^a A_\mu^b c^c$$

with $\delta_B K_a^\mu = \delta_B L_a = 0$ BRST inv. pure imaginary

- K_a^μ, L_a introduced by Zinn-Justin, B. Lee
  "anti-fields", covariant moment conjugate to A_μ^a, c^a
  anti-field formalism

derivation of Ward identity

$\mathcal{L}_{qu} + \mathcal{L}_{extra} + \mathcal{L}_{source}$ $\qquad$ $\mathcal{L}_{source} = J_a^\mu A_\mu^a + \underbrace{\underset{\downarrow\ \text{imaginary}}{\beta_a} c^a + b_a \underset{\downarrow\ \text{real}}{\gamma^a}}_{\text{anti-commuting}}$

path - integral

$$Z[J_a^\mu, \beta_a, \gamma^a; K_a^\mu, L_a)$$
$$= N \int \mathcal{D}A_\mu^a \mathcal{D}b_a \mathcal{D}c^a \, e^{\frac{i}{\hbar}\int d^4x \, (\mathcal{L}_{qu} + \mathcal{L}_{extra} + \mathcal{L}_{source})}$$

N is normalization s.t. $Z[0,0,0;0,0] = 1$

infinitesimal change of variables:

$$(A_\mu^a)' = A_\mu^a + \varepsilon \delta_B A_\mu^a$$
$$(b_a)' = b_a + \varepsilon \delta_B b_a$$
$$(c^a)' = c^a + \varepsilon \delta_B c^a$$

$\varepsilon \ll 1$ commuting para.
($\Lambda \ll 1$ is ambiguous)

- $\mathcal{D}A_\mu^a \mathcal{D}b_a \mathcal{D}c^a = \mathcal{D}A_\mu^{a\prime} \mathcal{D}b_a' \mathcal{D}c^{a\prime}$ by adding local counter terms
- BRST inv. $\quad \mathcal{L}_{qu} = \mathcal{L}(A_\mu^{a\prime}, b_a', c^{a\prime})$

$$\mathcal{L}_{extra} = K_a^\mu D_\mu c^{a\prime} + L_a \frac{1}{2} g f_{bc}{}^a c^{b\prime} c^{c\prime}$$

- replace $\mathcal{L}_{source} = J_\mu^a (A_\mu^{a\prime} - \delta_B A_\mu^a) + \beta_a (c^{a\prime} - \delta_B c^a) + (b_a' - \delta_B b_a)\gamma^a$

$$Z = \int \mathcal{D}A_\mu^{a'} \mathcal{D} b_a' \mathcal{D} c^{a'} \exp \frac{i}{\hbar}\Big[\int d^4x \big(\mathcal{L}_{gn}(A_\mu^{a'}, b_a', c^{a'}) + \mathcal{L}_{extra}(A_\mu^{a'}, c^{a'})$$

$$+ \mathcal{L}_{source}(A_\mu^{a'} - \delta_B A_\mu^a, c^{a'} - \delta_B c^a)\Big]$$

shakespeare theorem

$$= \int \mathcal{D}A_\mu^a \mathcal{D} b_a \mathcal{D} c^a \exp \frac{i}{\hbar}\Big[\int d^4x \big(\mathcal{L}_{gn}(A_\mu^a, b_a, c^a) + \mathcal{L}_{extra}(A_\mu^a, c^a)$$

$$+ \mathcal{L}_{source}(A_\mu^a - \delta_B A_\mu^a, c^a - \delta_B c^a)\Big]$$

$\Rightarrow$ up to $\mathcal{O}(\varepsilon')$: Ward identity

$$\int \mathcal{D}A_\mu^a \mathcal{D} b_a \mathcal{D} c^a \int d^4y \big(J_a^\mu(y)\, \delta_B A_\mu^a(y) + \beta_a(y)\, \delta_B c^a(y) + \delta_B b_a(y)\, \gamma^a(y)\big)$$

$$\exp \frac{i}{\hbar}\int d^4x\, (\mathcal{L}_{gn} + \mathcal{L}_{extra} + \mathcal{L}_{source}) = 0$$

In short,

$$\int d^4y \langle J_a^\mu \delta_B A_\mu^a + \beta_a \delta_B c^a + \delta_B b_a \gamma^a \rangle = 0$$

also note
$$\frac{i}{\hbar}\langle \delta_B A_\mu^a(y)\rangle = \left(\frac{\partial}{\partial K_a^\mu(y)} \Lambda\right) Z$$

$$\frac{i}{\hbar}\langle \delta_B c^a(y)\rangle = \left(\frac{\partial}{\partial L_a(y)} \Lambda\right) Z$$

reason we intro $\mathcal{L}_{extra}$

$$\frac{i}{\hbar}\langle \delta_B b_a\rangle = \left(-\frac{1}{\xi}\partial^\mu \frac{\partial}{\partial J_a^\mu}\Lambda\right) Z$$

because $\delta_B b_a = -\frac{1}{\xi}(\partial^\mu A_\mu^a)\Lambda$, $\quad \frac{i}{\hbar}\langle A_\mu^a\rangle = \frac{\partial}{\partial J_a^\mu} Z$

Ward identity simplifies to

$$\int d^4y \left(J_a^\mu \frac{\partial}{\partial K_a^\mu} + \beta_a \frac{\partial}{\partial L_a} + \frac{1}{3} \partial^\mu \frac{\partial}{\partial J_a^\mu} \gamma^a \right) Z = 0$$

linear 1st order PDE

- why $\frac{\partial}{\partial K_a^\mu} Z$ to get $\langle \delta_B A_\mu^a \rangle$ instead of $\frac{\partial}{\partial J_b^\mu(x)} \frac{\partial}{\partial \beta_c(x)} Z$?

  because the later has double derivative, ~~not~~ easy to perform Legendre transf.

Connected graphs

$$Z = e^{\frac{i}{\hbar} W} \qquad W = W[J_a^\mu, \beta_a, \gamma^a; K_a^\mu, L_a]$$

Ward identity $\int d^4y \left(J_a^\mu \frac{\partial}{\partial K_a^\mu} + \beta_a \frac{\partial}{\partial L_a} + \frac{1}{3} \left(\partial^\mu \frac{\partial}{\partial J_a^\mu} \right) \gamma^a \right) W = 0$

effective action

$$\Gamma(A_\mu^a, c^a, b_a; K_a^\mu, L_a) = W(J_a^\mu, \beta_a, \gamma^a; K_a^\mu, L_a) - \int (J_a^\mu A_\mu^a + \beta_a c^a + b_a \gamma^a) d^4x$$

tree level : $\Gamma^{tree} = S_{qn} + S_{extra}$

"Hamiltonian equation"

$\frac{\partial L}{\partial \dot{q}} = p$: $\quad \frac{\partial}{\partial J_a^\mu} W = A_\mu^a \quad \frac{\partial}{\partial \beta_a} W = c^a \quad \frac{\partial}{\partial \gamma^a} W = -b_a$

$\frac{\partial H}{\partial p} = \dot{q}$: $\quad \partial \Gamma / \partial A_\mu^a = -J_a^\mu, \quad \partial \Gamma / \partial c^a = -\beta_a \quad \partial \Gamma / \partial b_a = \gamma^a$

- notation : $\partial\Gamma/\partial A_\mu^a \equiv \Gamma \frac{\overleftarrow{\partial}}{\partial A_\mu^a}$, right derivative

  $\frac{\partial}{\partial A_\mu^a}\Gamma$: left derivative

  easy check, use $\Gamma = b_a \gamma^a$ as an example

- $A_\mu^a = \frac{\partial}{\partial J_a^\mu} W = \langle A_\mu^a \rangle_{\text{connected}}$: "classical fields"

  we are using same notation for classical fields to simplify derivation

In classical mechanics $\frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}$

similarly $\frac{\partial}{\partial K_a^\mu}\Gamma = \frac{\partial}{\partial K_a^\mu} W$ $\quad \frac{\partial}{\partial L_a}\Gamma = \frac{\partial}{\partial L_a} W$

Ward identity for Γ

$$\int d^4x \left[\left(-\partial\Gamma/\partial A_\mu^a(x)\right) \frac{\partial}{\partial K_a^\mu(x)}\Gamma + \left(-\partial\Gamma/\partial c^a(x)\right)\frac{\partial}{\partial L_a(x)}\Gamma + \frac{1}{\xi}\left(\partial_\mu A^{\mu a}(x)\right)\partial\Gamma/\partial b_a(x) \right] = 0$$

- tree level : Ward identity reduces to BRST inv.

- Ward identity for Γ is nonlinear in Γ, diff from other model (like linear sigma model)

- 2 terms w/ 2 Γ's , 1 term w/ 1 Γ

further simplification

using the fact that $\mathcal{L}_{fix}$ is not renormalized

define $\quad \Gamma = \hat{\Gamma} + \int d^4x \, \mathcal{L}_{fix}$

- the difference between Γ, $\hat{\Gamma}$ only at tree level
- $\mathcal{L}_{fix}$ has only A_μ^a, $\quad \frac{\partial}{\partial K_a^\mu(x)} \Gamma = \frac{\partial}{\partial K_a^\mu(x)} \hat{\Gamma}$, $\quad \frac{\partial}{\partial L_a(x)} \Gamma = \frac{\partial}{\partial L_a(x)} \hat{\Gamma}$

$$\partial \Gamma / \partial c^a = \partial \hat{\Gamma} / \partial c^a \qquad \partial \Gamma / \partial b_a = \partial \hat{\Gamma} / \partial b_a$$

Ward identity for Γ

$$\int d^4x \Big[\big(- \partial \Gamma / \partial A_\mu^a(x)\big) \frac{\partial}{\partial K_a^\mu(x)} \Gamma + \big(- \partial \Gamma / \partial c^a(x)\big) \frac{\partial}{\partial L_a(x)} \Gamma + \frac{1}{\xi} \big(\partial_\mu A^{\mu a}(x)\big) \, \partial \Gamma / \partial b_a(x) \Big] = 0$$

together with an identity

$$\int d^4x \Big[\big(- \partial (\Gamma - \hat{\Gamma}) / \partial A_\mu^a(x)\big) \frac{\partial}{\partial K_a^\mu(x)} \hat{\Gamma} + \frac{1}{\xi} \big(\partial^\mu A_\mu^a(x)\big) \, \partial \hat{\Gamma} / \partial b_a(x) \Big] = 0$$

Ward identity for $\hat{\Gamma}$

$$\int d^4x \Big[\big(\partial \hat{\Gamma} / \partial A_\mu^a(x)\big) \Big(\frac{\partial}{\partial K_a^\mu(x)} \hat{\Gamma} \Big) + \big(\partial \hat{\Gamma} / \partial c^a(x)\big) \Big(\frac{\partial}{\partial L_a(x)} \hat{\Gamma} \Big) \Big] = 0$$

"$\hat{\Gamma}\hat{\Gamma}$ - equation"

note

1. to prove

$$\int d^4x \left[\left(-\partial(\Gamma - \hat{\Gamma})/\partial A_\mu^a(x) \frac{\partial}{\partial K_a^\mu(x)} \hat{\Gamma} + \frac{1}{3}(\partial^\mu A_\mu^a(x)) \, \partial\hat{\Gamma}/\partial b_a(x) \right] = 0$$

Start with

$$\int \mathcal{D}A_\mu^a \mathcal{D}b_a \mathcal{D}c^a \frac{\partial}{\partial b_a(y)} e^{\frac{i}{\hbar}[S_{gu} + S_{extra} + S_{sources}]} = 0$$

because Grassman integral $\int db \, b = 1$, $\int db \, \underset{\text{no } b}{F} = 0$

$F(b) = F_0 + F_1 b$ because $b^2 = 0$

$\Rightarrow \frac{\partial}{\partial b} F(b) = F_1$ independent of b for arbitrary F

$$\frac{\partial}{\partial b_a(y)} S_{gu} = \partial^\mu D_\mu c^a(y) \qquad \frac{\partial}{\partial b_a(y)} S_{source} = \gamma^a(y)$$

$\Rightarrow \langle \partial^\mu D_\mu c^a(y) + \gamma^a(y) \rangle = 0$ local Ward identity

also $\frac{i}{\hbar} \langle D_\mu c^a(y) \rangle = \frac{\partial}{\partial K_a^\mu(y)} Z$

$$\Rightarrow \left(\partial^\mu \frac{\partial}{\partial K_a^\mu(y)} Z + \frac{i}{\hbar} \gamma^a(y) \right) Z = 0$$

divided by Z

$$\Rightarrow \partial^\mu \frac{\partial}{\partial K_a^\mu(x)} W + \gamma^a(x) = 0$$

$$\Rightarrow \partial^\mu \frac{\partial}{\partial K_a^\mu(x)} \Gamma - \frac{\partial}{\partial b_a(x)} \Gamma = 0 \Rightarrow \partial^\mu \frac{\partial}{\partial K_a^\mu(x)} \hat{\Gamma} - \frac{\partial}{\partial b_a(x)} \hat{\Gamma} = 0$$

- at $\mathcal{O}(\hbar^0)$, $\partial^\mu D_\mu c^a - \partial^\mu D_\mu c^a = 0$
- use $T - \hat{\Gamma} = S_{fix}$

$$\Rightarrow \quad -\int \partial S_{fix}/\partial A_\mu^a(x) \frac{\partial}{\partial K_a^\mu(x)} \hat{\Gamma}\, d^4x = \int \frac{1}{\xi}(\partial^\mu A_\mu^a)\, \partial^\mu \frac{\partial}{\partial K_a^\mu} \hat{\Gamma}\, d^4x$$

then we can prove the identity

2. prove self energy is always transversal

differentiate Ward identity for $\hat{\Gamma}$ w.r.t $A_\nu^b(y)$, $c^d(w)$, then set all remaining fields to zero

$$\frac{\partial^2}{\partial K \partial A} \hat{\Gamma} = \partial \hat{\Gamma}/\partial A = \frac{\partial}{\partial A} \partial \hat{\Gamma}/\partial c = \frac{\partial}{\partial L} \hat{\Gamma} = \partial \hat{\Gamma}/\partial c = 0 \cdots$$

after setting all remaining fields to zero, due to ghost # conservation or Lorentz invariance

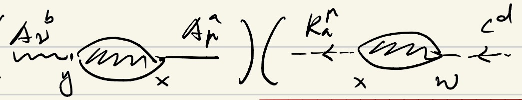
Ex: $\hat{\Gamma}$ has ghost # 0

$$\Rightarrow \partial \hat{\Gamma}/\partial c \sim b(\cdots) + KA(\cdots)$$

$\hat{\Gamma}$ is Lorentz inv.

$$\Rightarrow \partial \hat{\Gamma}/\partial A \sim A(\cdots) + \partial b(\cdots) + \partial c(\cdots)$$

$$\Rightarrow \int d^4x \left(\frac{\partial^2 \hat{\Gamma}}{\partial A_\mu^a(x)\, \partial A_\nu^a(y)} \right) \left(\frac{\partial^2 \hat{\Gamma}}{\partial K^\mu_a(x)\, \partial c^d(w)} \right) = 0$$

in graphs $\int d^4x \Big($ 
$\Big) = 0$

proportional to k^μ after Fourier transf.

$$\Rightarrow \quad k^\mu \langle A_\mu^a(k) A_\nu^b(-k) \rangle_{\hat\Gamma} = 0$$

$$\Gamma = \hat\Gamma + \underbrace{S_{fix}}_{\text{tree level}}$$

loop contribution to $\langle A_\mu^a(k) A_\nu^b(-k)\rangle$ comes from $\hat\Gamma$

transversality $\Rightarrow$ no contribution like $(\partial^\mu A_\mu^a)^2$ from loops

$\Rightarrow$ S_{fix} is not renormalized.

Summary: 2 Ward identities

I: $$\int d^4x \left[\left(\partial\hat\Gamma/\partial A_\mu^a(x)\right)\left(\frac{\partial}{\partial K_a^\mu(x)}\hat\Gamma\right) + \left(\partial\hat\Gamma/\partial c^a(x)\right)\left(\frac{\partial}{\partial L_a(x)}\hat\Gamma\right)\right] = 0$$

$\leftarrow \delta_B A_\mu^a, \ \delta_B c^a$

II $$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial K^\mu_a(x)} - \frac{\partial}{\partial b_a(x)}\right)\hat\Gamma = 0 \quad \leftarrow \delta_B b_a$$

- I non local, quadratic in $\hat\Gamma$, II local, linear in $\hat\Gamma$
- $b_a(x)$ dependence of $\hat\Gamma$ given by II
- other identities come from derivatives of these 2

$$S^{ren} = S(A_\mu^{a,ren}, b_a^{ren}, c^a_{ren}; K^{\mu,ren}_a, L_a^{ren}; u)$$

$$\Delta S^{ren} \equiv S - S^{ren} \qquad \text{with } S = S_{gu} + S_{extra}$$

also renorm. $K_a^\mu = \sqrt{Z_K}\, K_a^{\mu,ren}$, $L_a = \sqrt{Z_L}\, L_a^{ren}$

explicitly

$$S^{ren} = S_{gu}(A_\mu^{a,ren}, b_a^{ren}, c_{ren}^a, u)$$
$$+ \int d^4x \left[K_a^{\mu,ren}(\partial_\mu c_{ren}^a + u f_{bc}{}^a A_\mu^{b,ren} c_{ren}^c) + L_a^{ren} \tfrac{1}{2} u f_{bc}{}^a c^b_{ren} c^c_{ren} \right]$$

$$\Delta \mathcal{L}^{ren} = -\tfrac{1}{4}(Z_3 - 1)(\partial_\mu A_\nu^{a,ren} - \partial_\nu A_\mu^{a,ren})^2$$
$$- \tfrac{1}{4}(Z_1 - 1) u f_{bc}{}^a (\partial_\mu A_\nu^{a,ren} - \partial_\nu A_\mu^{a,ren}) A_\mu^{b,ren} A_\nu^{c,ren}$$
$$+ \cdots + (\sqrt{Z_L}\, Z_1 Z_{gh}/Z_3^{3/2} - 1) L_a^{ren} \tfrac{1}{2} u f_{bc}{}^a c^b_{ren} c^c_{ren}$$

- Γ computed using $S_{cl} + S_{fix} + S_{gh} + S_{extra}$ and unrenorm. fields

  Γ^{ren} computed using $S_{gu}^{ren} + S_{extra}^{ren} + \Delta S^{ren}$, renorm. fields

- divergences in Γ, $\varepsilon \to 0$

  if renorm. properly, $\lim_{\varepsilon \to 0} \Gamma^{ren}$ exists (keeping renorm. quantities fixed)

  ($\lim_{\varepsilon \to 0} \Gamma$ exists if one varies A_μ^a s.t. $A^a_{\mu,ren}$ fixed)

- for finite ε, $S = S^{ren} + \Delta S^{ren} \Rightarrow Z = Z^{ren} \Rightarrow \Gamma \overset{?}{=} \Gamma^{ren}$

  $J_a^\mu A_\mu^a + \cdots \qquad J_a^\mu A_\mu^{a,ren} + \cdots$

  one can prove $\Gamma = \Gamma^{ren}$

$\bullet \quad S_{fix} = S_{fix}^{ren} \Rightarrow \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 = \frac{1}{2\xi^{ren}}(\partial^\mu A_\mu^{a,ren})^2$

$$\Rightarrow Z_3 = Z_\xi \Rightarrow \hat{\Gamma} = \hat{\Gamma}^{ren}$$

Ward identities for $\hat{\Gamma}^{ren}$

$\hat{\Gamma}^{ren}$ is a finite functional of $A_\mu^{a,ren}$ ..., so $\frac{\partial\hat{\Gamma}}{\partial A_\mu^a(x)}$ ... also finite

II: $$\left[\partial_\mu \frac{1}{\sqrt{Z_K}}\frac{\partial}{\partial K_a^{\mu,ren}(x)} - \frac{1}{\sqrt{Z_{gh}}}\frac{\partial}{\partial b_a^{ren}(x)}\right]\hat{\Gamma}^{ren} = 0$$

$$\Rightarrow Z_K = Z_{gh} \quad \left(\frac{Z_K}{Z_g} = \text{finite} \qquad \begin{matrix} Z_K = 1 + \cdots \\ Z_g = 1 + \cdots \end{matrix} \right)$$

similarly, from I: $\quad Z_3 Z_K = Z_{gh} Z_L$

$$\Rightarrow Z_L = Z_3$$

Summarize all relations $\quad Z_3 = Z_\xi = Z_L, \quad Z_K = Z_{gh}.$

only 3 Z factors left: Z, Z_3, Z_{gh}, if renormalizable, no more than 3 independent divergences

renormalized Ward identities

I': $$\int d^4x \left[(\partial\hat{\Gamma}^{ren}/\partial A_\mu^{ren})\left(\frac{\partial}{\partial K_a^{\mu,ren}}\hat{\Gamma}^{ren}\right) + (\partial\hat{\Gamma}^{ren}/\partial c_{ren}^a)\left(\frac{\partial}{\partial L_a^{ren}}\hat{\Gamma}^{ren}\right)\right] = 0$$

II': $$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial K_a^{\mu,ren}(x)} - \frac{\partial}{\partial b_a^{ren}(x)}\right)\hat{\Gamma} = 0$$

$\text{II}' \Rightarrow \quad \hat{\Gamma}^{ren} = \hat{\Gamma}^{ren}(\partial^\mu b_a^{ren} - K_a^{\mu, ren})$

there is no way to find general solution to I', but divergent terms satisfies a simpler equation.

Induction:

1. assume the theory has renorm. up to $(n-1)$-loops.

$\Rightarrow \hat{\Gamma}^{ren}$ of order $\hbar^{n-1}$ and less are finite

$$\hat{\Gamma}^{ren} = \underbrace{\hat{S}^{ren} + \cdots + \hat{\Gamma}^{ren,(n-1)}_{finite}}_{finite} + \hat{\Gamma}^{ren(n)} + \cdots$$

$$\hat{S} = S - S_{fix}$$

2. $Z_3 = \tilde{Z}_3$, $Z_K = Z_{gh}$, $Z_L = Z_3$ holds upto $\mathcal{O}(\hbar^{n-1})$

$n-1=0$: 1 & 2 hold automatically

n: decompose $\hat{\Gamma}^{ren} = \hat{S}^{ren} + \cdots + \hat{\Gamma}^{ren,(n-1)}_{finite} + \hat{\Gamma}^{ren(n)}_{finite} + \hat{\Gamma}^{ren(n)}_{div} + \cdots$

compute $\hat{\Gamma}\hat{\Gamma}$ equation (I') at $\mathcal{O}(\hbar^n)$, $\hat{\Gamma}^{ren(n)}_{div}$ can only appear once in each $\hat{\Gamma}\hat{\Gamma}$ $(\hat{\Gamma}^{ren(n)}\hat{S}^{ren}$ or $\hat{S}^{ren}\hat{\Gamma}^{ren(n)})$

I' at $\mathcal{O}(\hbar^n)$:

$$\int d^4x \Big[\partial\hat{S}^{ren}/\partial A_\mu^{a,ren} \frac{\partial}{\partial K_a^{\mu,ren}} - \partial\hat{S}^{ren}/\partial K_a^{\mu,ren} \frac{\partial}{\partial A_\mu^{a,ren}}$$
$$+ \partial\hat{S}^{ren}/\partial c^a_{ren} \frac{\partial}{\partial L_a^{ren}} - \partial\hat{S}^{ren}/\partial L_a^{ren} \frac{\partial}{\partial c^a_{ren}} \Big] \hat{\Gamma}^{ren,(n)}_{div} = 0$$

from now on we drop "ren"

Slavnov - Taylor operator $\mathcal{S}'$

$$\mathcal{S}' = \int d^4x \left[\partial\hat{S}/\partial A_\mu^a \frac{\partial}{\partial K_a^\mu} - \partial\hat{S}/\partial K_a^\mu \frac{\partial}{\partial A_\mu^a} + \partial\hat{S}/\partial c^a \frac{\partial}{\partial L_a} - \partial\hat{S}/\partial L_a \frac{\partial}{\partial c^a} \right]$$

where $\hat{S} = S - S_{fix}$

$$\begin{cases} \mathcal{S}' A_\mu^a = -\partial\hat{S}/\partial K_a^\mu = D_\mu c^a = s A_\mu^a \\ \mathcal{S}' c^a = -\partial\hat{S}/\partial L_a = \frac{1}{2} f_{bc}{}^a c^b c^c = s c^a \end{cases}$$

$\mathcal{S}'$ on A_μ^a, c^a generates BRST transf.

- $\mathcal{S}'$ is not quite BRST change, because $\mathcal{S}' K_a^\mu$, $\mathcal{S}' L_a \neq 0$
- $\mathcal{S}'$ is independent of $\hbar$
- $\mathcal{S}'$ is nilpotent !

proof: set $x^i = \{A_\mu^a, L_a\}$, $\theta_i = \{K_a^\mu, -c^a\}$

$$\mathcal{S}' = \left(\partial\hat{S}/\partial x^i \frac{\partial}{\partial\theta_i} - \partial\hat{S}/\partial\theta_i \frac{\partial}{\partial x^i} \right)$$

use $\frac{\partial}{\partial x^i}\hat{S}\frac{\partial}{\partial\theta_i}\hat{S} = 0$ (why?), one can show $\mathcal{S}'^2 = 0$

Conclusion

BRST: $\mathcal{S}'^{\,ren}\, \hat{\Gamma}^{ren,(n)}_{div} = 0$, $(\mathcal{S}'^{\,ren})^2 = 0$

$\hat{\Gamma}^{ren,(n)}_{div}$ satisfies a linear PDE.

multiplicative renormalizability of pure YM

$(S^{\text{ren}})^2 = 0$, ansatz for $\hat{\Gamma}^{(n)}_{div} = \alpha S_{cl} + S^{\text{ren}} X$

- S_{cl} is any gauge inv. action
- X is any Lorentz inv, group inv. polynomial with correct dim and ghost #
- α, parameters in X have the form

$$\hbar^n \mu^{2n} \left(\frac{1}{\varepsilon^n} \underline{C_n} + \cdots + \frac{1}{\varepsilon} \underline{C_1} \right)$$

#, group inv: $C_2(R)$, $T(R)$

dispersion relations (unitarity) $\Rightarrow$ $\hat{\Gamma}^{(n)}_{div}$ is local

to show the ansatz is most general:

① cohomology of Lie algebra or ② power counting for proper gr.

power counting method

steps

i) determine the set of all proper graphs which could be div. from power counting

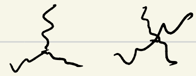
ii) narrow the set down by require they are S-closed

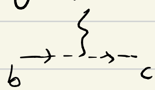
iii) show the remaining set of div. is as the ansatz

possible terms from power counting

a graph: ℓ: independent 4-momenta loops

I_A / I_{bc}: internal YM / ghost propagators

n_j: vertices of gauge fields $j = 3, 4$ 

n_{bAc}: ghost vertices 

n_K: vertices of KAc (from $K_a^\mu D_\mu c^a$)

n_L: vertices of Lcc (from $L_a \frac{1}{2} f_{bc}{}^a c^b c^c$)

E_b: external b_a

degree of div.

$$D = 4\ell - 2I_A - 2I_{bc} + \overset{\overset{\partial}{\uparrow}}{\underline{n_3}} + \overset{\overset{\partial}{\uparrow}}{\underline{n_{bAc}}} - \overset{\overset{b \text{ in } \mathcal{L}_{gf} \text{ as } \partial b}{\uparrow}}{\underline{E_b}}$$

also $\ell = I_A + I_B - n_3 - n_4 - n_{bAc} - n_{KAc} - n_L + 1$

(Euler formula)

and. $E_A + 2I_A = 3n_3 + 4n_4 + n_{bAc} + n_{KAc}$

$$E_b + E_c + 2I_{bc} = 2n_{bAc} + n_{KAc} + 2n_L$$

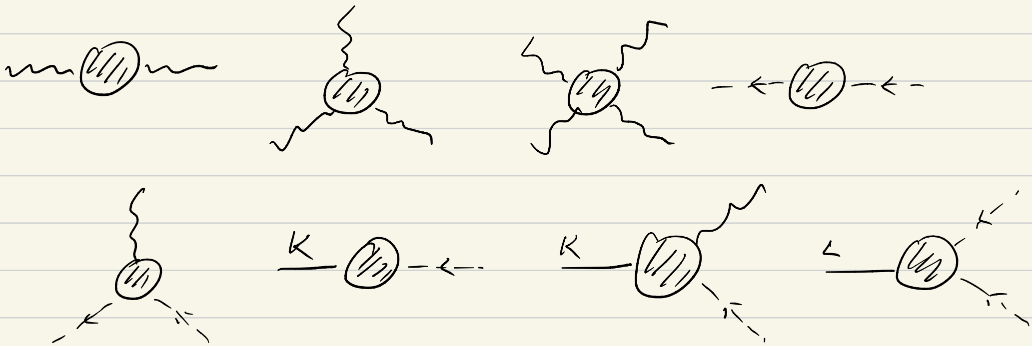
$$\Rightarrow D = 4 - E_A - 2E_b - E_c - 2n_{KAc} - 2n_L$$

- 5 and more external lines, $D < 0$, safe
- potential divergence from power counting

$$A^4, \quad \partial A^3, \quad \partial^2 A^2, \quad \partial^2 bc, \quad \partial bAc$$

$$\partial Kc \quad KAc, \quad Lcc$$

derivatives can distribute arbitrarily, contracting Lorentz and group indices to give scalars



- vacuum bubbles cancelled by overall normalization of Z
- tadpole graphs vanish : no fields have the quantum number of the vacuum $\Rightarrow \langle A_\mu^a \rangle = \langle b_a \rangle = \langle c^a \rangle = 0$

  otherwise the vac. is not Lorentz / gauge inv.
- no $b^2 c^2$ because always ∂b

In the end:

$$\Gamma^{\text{ren}(n)}_{\text{div}} = \int d^4x \Big[(A^4 + \partial A^3 + \partial^2 A^2) + (K_a^\mu - \partial^\mu b_a)(a\,\partial_\mu c^a + b\, g_{bc}{}^a A_\mu^b c^c) + \tfrac{1}{2} c\, h_{bc}{}^a L_a c^b c^c \Big]$$

- $g_{bc}{}^a$, $h_{bc}{}^a$: inv tensor of gauge group
- coeffs. are possibly divergent, much more terms than 3 needs constraint from S
- $a, b, c \sim \frac{1}{n-1} \hbar^n u^{2n}$, coeff of $A^2 \cdots \sim \frac{1}{n-1}$

S - closeness

$$S = \int d^4x \Big[\Big(\frac{\partial}{\partial A_\mu^a} \hat{S}\Big) \frac{\partial}{\partial K_a^\mu} + D_\mu c^a \frac{\partial}{\partial A_\mu^a} - \Big(\frac{\partial}{\partial c^a} \hat{S}\Big) \frac{\partial}{\partial L_a} - \tfrac{1}{2} u f_{bc}{}^a c^b c^c \frac{\partial}{\partial c^c} \Big]$$

Solving $\mathcal{S}\,\hat{\Gamma}^{(n)}_{div} = 0$

many possible contractions for $A^4 + \partial A^3 + \partial^2 A^2$

look at terms are not $\partial c(A^3 + \partial A^2 + \partial^2 A)$ or $Ac(A^3 + \partial A^2 + \partial^2 A)$ because $\mathcal{S}(A^4 + \partial A^3 + \partial^2 A^2)$ produces these.

Lcc terms

$$0 = \int d^4x \left[(\partial \hat{S}/\partial c^a)\frac{\partial}{\partial L_a} - (\partial \hat{S}/L_a)\frac{\partial}{\partial c^a}\right]\hat{\Gamma}^{(n)}_{div}$$

$$= \int d^4x \left[(\partial \hat{S}/\partial c^a)\tfrac{1}{2}\gamma h_{bc}{}^a c^b c^c - \gamma L_a h_{bc}{}^a \left(\tfrac{u}{2} f_{pq}{}^b c^p c^q\right) c^c \right]$$

$$= \int d^4x \left[L_a u f_{bc}{}^a c^b \left(\tfrac{1}{2}\gamma h_{pq}{}^c c^p c^c\right) - \gamma L_a \tfrac{u}{2} h_{bc}{}^a f_{pq}{}^b c^p c^q c^c\right.$$

$$\Rightarrow f_{bs}{}^a h_{pq}{}^s c^b c^p c^q = h_{sb}{}^a f_{pq}{}^s c^p c^q c^b$$

$$\Rightarrow h_{pq}{}^s = \alpha f_{pq}{}^s + \beta \underbrace{d_{pq}{}^s}_{\text{anomaly coeff.}}$$

however $h_{pq}{}^s = -h_{qp}{}^s \Rightarrow h_{pq}{}^s = \alpha f_{pq}{}^s$

we absorb α in coefficient, $h_{pq}{}^s = f_{pq}{}^s$

$K\partial cc$, $KAcc$ terms

$$0 = \int d^4x \left[(\partial \hat{S}/\partial A^a_\mu)\frac{\partial}{\partial K^\mu_a} + (\partial \hat{S}/\partial c^a)\frac{\partial}{\partial L_a} + (D_\mu c)^a \frac{\partial}{\partial A^a_\mu} - \tfrac{1}{2} u f_{bc}{}^a c^b c^c \frac{\partial}{\partial c^a}\right]\hat{\Gamma}^{(n)}$$

$$= \int d^4x \left[\underline{K^\mu_a u f_{bc}{}^a (\alpha \partial_\mu c^b} + \beta g_{pq}{}^b A^p_\mu c^q)\underline{c^c} + K^\mu_a \gamma f_{bc}{}^a (D_\mu b^b) c^c \right.$$

$$\left. + K^\mu_a \beta g_{bc}{}^a (D_\mu c^b) c^c + \underbrace{K^\mu_a(\alpha(-u) f_{bc}{}^a (\partial_\mu c^b) c^c}_{\text{cancel}} + \beta g_{pq}{}^a A^p_\mu (-\tfrac{1}{2} u f_{rs}{}^q c^r c^s)\right]$$

$$\Rightarrow \quad \gamma f_{bc}{}^{a} + \beta g_{bc}{}^{a} = 0$$

$$f_{bc}{}^{a} g_{pq}{}^{b} c^{q} c^{c} - \frac{1}{2} g_{pq}{}^{a} f_{rs}{}^{q} c^{r} c^{s} = 0$$

$$\Rightarrow \quad g_{bc}{}^{a} = f_{bc}{}^{a}$$

$\downarrow$ Jacobi identity

$A^4 + \partial A^3 + \partial^2 A^2$ term

$(D_\mu c)^a \frac{\partial}{\partial A_\mu^a}$ in S gains contribution from these terms

$$\Rightarrow \int d^4x \left[(D_\mu c)^a \frac{\partial}{\partial A_\mu^a} \left(\int A^4 + \partial A^3 + \partial^2 A^2 \right) + \frac{\partial S_{YM}}{\partial A_\mu^a} \left(a \partial_\mu c^a - \gamma f_{bc}{}^{a} A_\mu^b c^c \right) \right] = 0$$

firstly gauge inv of S_{YM} (BRST) $\Rightarrow \int d^4x \frac{\partial S_{YM}}{\partial A_\mu^a} (D_\mu c)^a = 0$

$\Rightarrow$ we can replace $-\gamma f_{bc}{}^{a} A_\mu^b c^c$ with $\gamma \partial_\mu c^a$ in eq.

$$\Rightarrow \int d^4y \, (D_\mu c)^a \frac{\partial}{\partial A_\mu^a} \left(\int d^4x \left(A^4 + \partial A^3 + \partial^2 A^2 \right) \right) + (a+\gamma) \int d^4y \frac{\partial S_{YM}}{\partial A_\mu^a} \partial_\mu c^a = 0$$

general solution is

$$\int d^4x \left(A^4 + \partial A^3 + \partial^2 A \right) = F + \alpha S_{YM}$$

αS_{YM} is the solution of the homogeneous eq.

$$\int d^4y \, (D_\mu c)^a \frac{\partial}{\partial A_\mu^a} \left(\int d^4x \left(A^4 + \partial A^3 + \partial^2 A^2 \right) \right) = 0$$

F is a particular solution of the original eq.

claim: $F = -(a+\gamma)\int d^4x\, A^b_\nu(x) \frac{\partial}{\partial A^b_\nu(x)} S_{YM}$

check let $\vartheta_1 = \int d^4y\, D_\mu c^a(y) \frac{\partial}{\partial A^a_\mu(y)}$, $\vartheta_2 = \int d^4x\, A^b_\nu(x) \frac{\partial}{\partial A^b_\nu(x)}$

$$F = -(a+\gamma)\,\vartheta_2\, S_{YM}$$

eq. is $\vartheta_1 F + (a+\gamma)\int d^4x \frac{\partial S_{YM}}{\partial A^a_\mu(x)} \partial_\mu c^a(x) = 0$

$$\Rightarrow -(a+\gamma)\,\vartheta_1 \vartheta_2 S_{YM} + (a+\gamma)\int d^4x \frac{\partial S_{YM}}{\partial A^a_\mu(x)} \partial_\mu c^a(x) = 0$$

because $\vartheta_1 S_{YM} = 0$

$$\Rightarrow -(a+\gamma)[\vartheta_1, \vartheta_2] S_{YM} + (a+\gamma)\int d^4x \frac{\partial S_{YM}}{\partial A^a_\mu(x)} \partial_\mu c^a(x) = 0$$

one can check explicitly $[\vartheta_1, \vartheta_2] = \int d^4x\, \partial_\mu c^a(x) \frac{\partial}{\partial A^a_\mu}$

$\Rightarrow$ $F = -(a+\gamma)\,\vartheta_2\, S_{YM}$ is a solution

The solution of $\mathcal{S}\Gamma^{(n)}_{div} = 0$

put all together, and $\beta = -(a+\gamma)$, $c = \gamma$

$$\hat{\Gamma}^{(n)}_{div} = \alpha S_{YM} + \beta \int d^4x\, A^b_\nu \frac{\partial}{\partial A^b_\nu} S_{YM}$$

$$+ \int d^4x\, (K^\mu_a - \partial^\mu b_a)\left[(\gamma-\beta)\partial_\mu c^a + \gamma u f_{bc}{}^a A^b_\mu c^a\right]$$

$$+ \gamma \int d^4x\, L_a \frac{1}{2} u f_{bc}{}^a c^b c^a$$

- only 3 parameters α, β, γ
- again, no divergences proportional to $\mathcal{L}_{fix}$, consistent
- $\hat{\Gamma}^{(n)}_{div} = \alpha S_{YM} + \underline{\mathcal{S}} X$ → dim 3, ghost # −1, Lorentz inv.

Lorentz inv, ghost # 1, dim 1

$$X = A\int d^4x\,(\partial^\mu b_a - K_a^\mu)A_\mu^a \qquad + B\int d^4x\, L_a c^a$$

compare $\mathcal{S}X$ with the solution to fix A, B

$(A = -\beta,\ B = \gamma)$

Absorbing divergences

- good sign: # of parameters in div (α, β, γ)

= # of renorm. para (Z, Z_3, Z_{gh})

$$A_\mu^{a\,(n-1)} = \sqrt{\frac{Z_3^{(n)}}{Z_3^{(n-1)}}}\,A_\mu^{a(n)} = \left(1 + \frac{1}{2} Z_3 \hbar^n + \cdots\right) A_\mu^{a(n)}$$

for $\phi = b_a,\ c^a,\ K_a^\mu$

$$\phi^{(n-1)} = \left(1 + \frac{1}{2} Z_{gh} \hbar^n + \cdots\right)\phi^{(n)}$$

finally,
$$u^{(n-1)} = \left(1 + \left(Z_1 - \frac{3}{2} Z_3\right)\hbar^n + \cdots\right)u^{(n)}$$

plug them in $S^{(n-1)}$, and keep terms up to $\hbar^n$

$$S^{(n-1)} = S^{(n)} + \underline{\text{terms linear in } Z\text{'s}}$$

↓

counter terms to cancel $\Gamma^{(n)}_{div}$

$\mathcal{L}_{gh} + \mathcal{L}_{extra}$

$$Z_{gh}(K_a^\mu - \partial^\mu b_a)\partial_\mu c^a + \underline{(Z_{gh}+Z_1-Z_3)}(K_a^\mu - \partial^\mu b_a)\, u f_{bc}{}^a A_\mu^b c^c$$

$$+ \underline{(Z_{gh}+Z_1-Z_3)}\, L_a \tfrac{1}{2} u f_{bc}{}^a c^b c^c$$

these 2 coeff. equal due to BRST

cancells ghost dependent terms in $\Gamma^{(n)}_{div}$ if

$$Z_{gh} = \beta - \gamma, \qquad Z_{gh} + Z_1 - Z_3 = -\gamma$$

$\mathcal{L}_{YM}$

$$Z_3 (\partial A)^2 + Z_1 u (\partial A) A^2 + (2Z_1 - Z_3) u^2 A^4$$

cancells div in $\alpha S_{YM} + \beta \int A_\mu^a \frac{\partial}{\partial A_\mu^a} S_{YM}$ if

$$Z_3 = -(\alpha + 2\beta)$$

$$Z_1 = -(\alpha + 3\beta)$$

$$2Z_1 - Z_3 = -(\alpha + 4\beta) \qquad \underline{Z_1 - Z_3 = -\beta}$$

consistent

3 eq. but 2 independent variable due to BRST

This concludes our proof of the renormalizability of pure YM theory

multiplicative renormalizability of quarks and gluons

classical action of Dirac fermions

$$\mathcal{L}_{fer} = -\bar{\psi}_i \gamma^\mu (D_\mu \psi)^i - m \bar{\psi}_i \psi^i$$

where $(D_\mu \psi)^i = \partial_\mu \psi^i + g A_\mu^a (T_a)^i{}_j \psi^j$

$$\bar{\psi}_i = (\psi^i)^\dagger i \gamma^0$$

$$\{\gamma_\mu , \gamma_\nu\} = 2\eta_{\mu\nu} \qquad (\gamma_\mu)^\dagger = \gamma^\mu$$

gauge transformation

$$\begin{cases} \delta_{gauge} \psi^i = -g (T_a)^i{}_j \psi^j \lambda^a(x) \\ \delta_{gauge} \bar{\psi}_i = g \bar{\psi}_j (T_a)^j{}_i \lambda^a(x) \end{cases}$$

$$[T_a , T_b] = f_{ab}{}^c T_c \qquad T_a^\dagger = -T_a$$

BRST transf.

$$\begin{cases} \delta_B \psi^i = -g (T_a)^i{}_j \psi^j c^a \Lambda \\ \delta_B \bar{\psi}_i = g \bar{\psi}_j (T_a)^j{}_i c^a \Lambda \end{cases}$$

Introduce $\mathcal{L}_{extra}$, $\mathcal{L}_{source}$ for fermions

$$\mathcal{L}_{new,f} = \mathcal{L}_f - g \underline{\bar{N}}\, T_a \psi c^a - g \bar{\psi}\, T_a \underline{N} c^a + \underline{\bar{J}_i} \psi^i + \bar{\psi}_i \underline{J^i}$$

commuting, gh # -1   anticommuting

$$\bar{N}_j = (N^j)^\dagger i \gamma^0 \qquad \bar{J}_i = (J^i)^\dagger i \gamma^0$$

Ward identity for $\hat{\Gamma}$

$$\left(\frac{\partial}{\partial b_a} - \partial^\mu \frac{\partial}{\partial K^a_\mu}\right)\hat{\Gamma} = 0 \rightarrow$$ unchanged because $\mathcal{L}_{fix}$, $\mathcal{L}_{ghost}$ don't depend on $\psi, \bar{\psi}$

$$\int d^4x \Big[(\partial\hat{\Gamma}/\partial A^a_\mu) \frac{\partial}{\partial K^\mu_a}\hat{\Gamma} + (\partial\hat{\Gamma}/\partial c^a)\frac{\partial}{\partial L_a}\hat{\Gamma}$$

$$\underbrace{+ (\partial\hat{\Gamma}/\partial\psi^i)\frac{\partial}{\partial\bar{N}_i}\hat{\Gamma} + (\partial\hat{\Gamma}/\partial N^i)\frac{\partial}{\partial\bar{\psi}_i}\hat{\Gamma}}_{\text{new terms from fermions}} \Big] = 0$$

question: show $\mathcal{L}_{fix}$ is not renormalized?

Z factors

$\psi = \sqrt{Z_f}\,\psi^{ren}$, $\bar{\psi} = \sqrt{Z_f}\,\bar\psi^{ren}$, $N = \sqrt{Z_N}\,N^{ren}$, $\bar{N} = \sqrt{Z_N}\,\bar{N}^{ren}$

only $\bar{\psi}\psi$, $\bar{N}\psi$, $\bar{\psi}N$ pair enter, so choose $Z_\psi = Z_{\bar\psi} = Z_f$

$$\begin{cases} \mathcal{L}_{fix} \Rightarrow Z_3 = Z_3 \\ \text{local WI} \Rightarrow Z_{gh} = Z_K \\ \Gamma\Gamma \Rightarrow Z_3 Z_K = Z_{gh} Z_2 = Z_f Z_N \end{cases} \Rightarrow \begin{cases} Z_N = \dfrac{Z_3 Z_{gh}}{Z_f} \\ Z_b = Z_L \end{cases}$$

mass: $m = Z_m\, m^{ren}$

independent Z factors: Z_3, Z_1, Z_{gh}, Z_f, Z_m

Slavnov – Taylor operator

$$\mathcal{S} = \mathcal{S}_{pure} + \mathcal{S}_f$$

$$\mathcal{S}_{pure} = \int d^4x \left[\left(\frac{\partial}{\partial A^a_\mu}\hat{S}\right)\frac{\partial}{\partial K^\mu_a} + (D_\mu c)^a \frac{\partial}{\partial A^a_\mu} - \left(\frac{\partial}{\partial c^a}\hat{S}\right)\frac{\partial}{\partial L_a} - \left(\frac{1}{2}u f_{bc}{}^a c^b c^c\right)\frac{\partial}{\partial c^a}\right]$$

$$\mathcal{S}_f = \int d^4x \left[(\partial\hat{S}/\partial\psi^i)\frac{\partial}{\partial \bar{N}_i} + (\partial\hat{S}/\partial N^i)\frac{\partial}{\partial\bar{\psi}_i} - \left(\frac{\partial}{\partial\bar{N}_i}\hat{S}\right)\frac{\partial}{\partial\psi^i} + \left(\frac{\partial}{\partial\bar{\psi}_i}\hat{S}\right)\frac{\partial}{\partial N^i}\right]$$

where $\hat{S} = S - S_{fix}$

Similarly $\mathcal{S}^2 = 0$,

Solution of $\mathcal{S}\hat{\Gamma}^{(n)}_{div} = 0$

$$\hat{\Gamma}^{(n)}_{div} = \alpha_1 S_{YM} + \alpha_2 S_{Dirac} + \alpha_3 S_{mass} + \mathcal{S}X$$

$$X = \int d^4x \left[\beta(K_a^\mu - \partial^\mu b_a)A^a_\mu + \gamma L_a c^a + \delta\bar{N}_i\psi^i + \varepsilon\bar{\psi}_i N^i\right]$$

note :

- power counting

$$D = 4 - E_A - 2E_b - E_c - 2n_K - 2n_L - \frac{3}{2}(n_N + n_{\bar{N}} + E_\psi + E_{\bar{\psi}})$$

now possible divergences in $\hat{\Gamma}$:

$$\int \bar{\psi}\partial\psi , \quad \int\bar{\psi}A\psi , \quad \int M\bar{\psi}\psi , \quad \int\bar{N}\psi c , \quad \int\bar{\psi}Nc$$

- 7 divergences (α_1, α_2, α_3, β, γ, δ, ε) vs 5 Z factors ?

evaluating δX:

$$\hat{\Gamma}^{(n)}_{div} = \alpha_1 S_{YM} + \alpha_2 S_{Dir} + \alpha_3 S_{mass} + \underline{\beta A^a_\mu \frac{\partial}{\partial A^a_\mu} \hat{S}}$$
$$\underline{- \beta}(K^\mu_a \underline{- \partial^\mu b_a)(D_\mu c)^a} + \gamma\left(c^a \frac{\partial}{\partial c^a}\hat{S} - L_a \frac{\partial}{\partial L_a}\hat{S}\right)$$
$$+ \delta\left(\psi^i \frac{\partial}{\partial \psi^i} - \bar{N}_i \frac{\partial}{\partial \bar{N}_i}\right)\hat{S} - \varepsilon\left(\bar{\psi}_i \frac{\partial}{\partial \bar{\psi}_i} - N^i \frac{\partial}{\partial N^i}\right)\hat{S}$$

$$= \alpha_1 S_{YM} + \alpha_2 S_{Dir} + \alpha_3 S_{mass} + \underline{\beta A^a_\mu \frac{\partial}{\partial A^a_\mu} S_{YM}}$$
$$+ (\gamma - \beta)(K^\mu_a - \partial^\mu b_a)\partial_\mu c^a + \gamma(K^\mu_a - \partial^\mu b_a) u f_{bc}{}^a A^b_\mu c^c$$
$$+ \gamma L_a \frac{1}{2} u f_{bc}{}^a c^b c^c + \gamma u(-\bar{N} T_a \psi c^a - \bar{\psi} T_a N c^a)$$
$$+ \left(-\varepsilon \bar{\psi}_i \frac{\partial}{\partial \bar{\psi}_i} + \delta \psi^i \frac{\partial}{\partial \psi^i}\right)(S_{Dir} + S_{mass})$$

notice $\left(-\varepsilon \bar{\psi}_i \frac{\partial}{\partial \bar{\psi}_i} + \delta \psi^i \frac{\partial}{\partial \psi^i}\right)(S_{Dir} + S_{mass}) = (-\varepsilon + \delta)(S_{Dir} + S_{mass})$

$\Rightarrow$

$$\hat{\Gamma}^{(n)}_{div} = \alpha_1 S_{YM} + \underbrace{(\alpha_2 - \varepsilon + \delta)}_{\alpha_2'} S_{Dir} + \underbrace{(\alpha_3 - \varepsilon + \delta)}_{\alpha_3'} S_{mass} + \beta A^a_\mu \frac{\partial}{\partial A^a_\mu} S_{YM}$$
$$+ (\gamma - \beta)(K^\mu_a - \partial^\mu b_a)\partial_\mu c^a + \gamma(K^\mu_a - \partial^\mu b_a) u f_{bc}{}^a A^b_\mu c^c$$
$$+ \gamma L_a \frac{1}{2} u f_{bc}{}^a c^b c^c + \gamma u(-\bar{N} T_a \psi c^a - \bar{\psi} T_a N c^a)$$

In the end, 5 divergences α_1 α_2' α_3' β, γ

5 Z factors Z_1 Z_3 Z_{gh} Z_f Z_m

absorb divergences with Z - factors

$$\psi^{i,\,ren,\,(n-1)} = \sqrt{\frac{Z_f^{(n)}}{Z_f^{(n-1)}}}\,\psi^{i,\,ren,\,(n)} = \left(1+\frac{1}{2}\hbar^n z_f + \cdots\right)\psi^{i,\,ren,\,(n)}$$

$$\Rightarrow \begin{cases} \underline{z_f + \alpha_2' = 0} \\ z_m + z_f + \alpha_3' = 0 \\ \underline{z_1 - z_3 + z_f + \alpha_2' + \beta = 0} \\ z_1 - z_3 + z_{gh} + \gamma = 0 \end{cases} \quad \Rightarrow \quad z_1 - z_3 + \beta = 0$$

Dirac fields minimally coupled to gauge fields are renormalizable.

chiral interaction : $g\bar{\psi}_i \gamma_5 \gamma^\mu A_\mu^a (T^a)_{ij} \psi^j$

possible anomaly in BRST Jacobian

$$J \sim \mathrm{Tr}\, T_a c^a (1+\gamma_5) e^{\not{D}\not{D}}$$

- $\Gamma\Gamma$ - eq.

$$(\hat{\Gamma},\hat{\Gamma}) = \Delta \quad . \quad (\hat{\Gamma},\hat{\Gamma}) : \text{antifield bracket} .$$

and $(\hat{\Gamma},\Delta) = 0$

- if Δ is BRST exact, it can be removed by a finite local counter term
- if Δ is not BRST exact (e.g. chiral anomaly), the theory is nonrenormalizable

1-loop β-function and asymptotic freedom

1-loop Z factors

In Lorentz gauge, up to 1-loop

$$Z_3 = 1 + \frac{10}{3} C_2(G)\, y - \frac{8}{3} T(R)\, y + (1-\xi) C_2(G)\, y$$

$$Z_1 = 1 + \left(\frac{4}{3} C_2(G) - \frac{8}{3} T(R)\right) y + (1-\xi)\frac{3}{2} C_2(G)\, y \quad \cdots$$

where $y = \frac{u^2}{16\pi^2(4-n)}$

and $C_2(G)$: 2nd Casimir $\qquad f_{pa}{}^{q} f_{qb}{}^{p} = -C(R)\delta_{ab}$

i.e. $\delta_{ab} C(R) = -\mathrm{Tr}(T_a^{ad} T_b^{ad})$

$T(R)$: $\mathrm{Tr}(T_a^R T_b^R) = -T(R)\delta_{ab}$

for $SU(N)$, $C_2(G) = N$, $T(\square) = \frac{1}{2}$

running coupling

$$g = \frac{Z_1}{Z_3^{3/2}} \mu^{2-\frac{d}{2}} u$$

using the fact that g is independent of the renormalization scale μ

$$0 = \mu \frac{d}{d\mu} g = \mu^{2-\frac{d}{2}} u\, \mu \frac{d}{d\mu}\left(\frac{Z_1}{Z_3^{3/2}}\right) + \frac{Z_1}{Z_3^{3/2}}\left(2-\frac{d}{2}\right)\mu^{2-\frac{d}{2}} u$$

$$+ \frac{Z_1}{Z_3^{3/2}} \mu^{2-\frac{d}{2}} \mu \frac{du}{d\mu}$$

$$\Rightarrow \mu \frac{du}{d\mu} = -\left(2-\frac{d}{2}\right) u - u \cdot \left(\frac{Z_1}{Z_3^{3/2}}\right)^{-1} \mu \frac{d}{d\mu}\left(\frac{Z_1}{Z_3^{3/2}}\right)$$

$$= -\left(2-\frac{d}{2}\right) u - u \mu \frac{d}{d\mu} \ln \frac{Z_1}{Z_3^{3/2}}$$

Z_1 and Z_3 depend on u but not on μ

$$\Rightarrow \mu \frac{du}{d\mu} = -\left(2-\frac{d}{2}\right) u - u \mu \frac{du}{d\mu} \frac{d}{du} \ln \frac{Z_1}{Z_3^{3/2}}$$

define β-function : $\beta(u) \equiv \mu \frac{du}{d\mu}$,

$$\beta(u) = \mu \frac{d\mu}{d\mu} = \left(\frac{d}{2}-2\right) u - u \beta(u) \frac{d}{du} \ln \frac{Z_1}{Z_3^{3/2}}$$

$$\Rightarrow \beta(u) = \frac{\frac{1}{2}(d-4) u}{1 + u \frac{d}{du} \ln \frac{Z_1}{Z_3^{3/2}}}$$

and $\frac{Z_1}{Z_3^{3/2}} = 1 - by = 1 - b \frac{u^2}{16\pi^2} \frac{1}{4-d}$

$$b = \frac{11}{3} C_2(A) - \frac{4}{3} T(R_F) - \frac{1}{3} T(R_S)$$

rep of complx fermions

rep of complx scalars

$$\lim_{d \to 4} \beta(u) = -\frac{u^3}{16\pi^2} b$$

- RG equation $\mu \frac{d}{d\mu} u(\mu) = \beta(\mu)$ governs how coupling runs with renormalization scale
- only for non-Abelian gauge theory, b is possible to be positive

define $\alpha \equiv \frac{u^2}{4\pi}$ (similar to fine structure const $\frac{e^2}{4\pi}$)

$$\mu^2 \frac{d^2}{d\mu^2} \frac{1}{\alpha} = \frac{b}{4\pi}$$

$$\Rightarrow \frac{1}{\alpha(M^2)} - \frac{1}{\alpha(\mu^2)} = \frac{b}{4\pi} \ln \frac{M^2}{\mu^2}$$

if $b > 0$ $\quad \lim_{\mu \to \infty} \alpha(\mu^2) = 0$ asymptotic freedom

(Gross, Wilczek / Politzer)

in QED, $b < 0$ $\quad \lim_{\mu \to \infty} \alpha(\mu^2) = \infty$ IR freedom

anti-screening

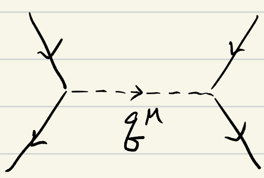
Coulomb gauge here : $f^a = \partial^i A_i^a$ $\quad i = 1, 2, 3$

$\mathcal{L}_{ghost} = b_a \partial^i (D_i c)^a$ $\quad$ ghosts won't couple with A_0^a

generalization of ϕ in QED)

two on-shell fermions exchanging Coulomb gluons (A_0^a)

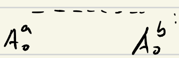
tree level



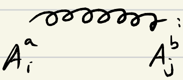
in center of mass frame

$q^0 = 0 \qquad q^\mu = (0, \vec{q})$

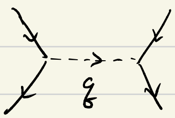
• propagators in Coulomb gauge



$$\langle A_0^a(\vec{k})\, A_0^b(-\vec{k})\rangle = \frac{-i\,\eta_{00}\,\delta^{ab}}{\vec{k}^2} = \frac{i}{\vec{k}^2}\,\delta^{ab}$$



$$\langle A_i^a(\vec{k})\, A_j^b(-\vec{k})\rangle = \frac{-i}{k^2 - i\varepsilon}\,\underbrace{P_{ij}(\vec{k})}_{\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}}\,\delta^{ab}$$



$$\sim \frac{1}{\vec{q}^2}$$

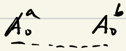
• Fourier transf of $\frac{1}{\vec{q}^2}$

$q = |\vec{q}|$

$$\int d^4q\, \frac{1}{\vec{q}^2}\, e^{i q_\mu x^\mu} = \int \frac{1}{q^2}\, e^{i q r\cos\theta}\, q^2\, dq\, d\cos\theta\, d\varphi$$

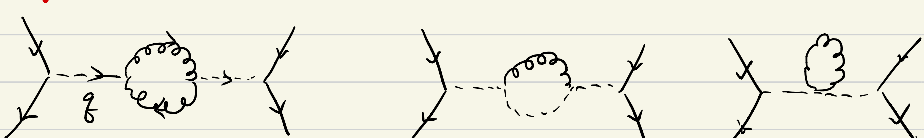
$$= \int e^{iqr\cos\theta}\, dq\, d\cos\theta\, d\varphi = 4\pi \int_0^\infty \frac{1}{iqr}\left(e^{-iqr} - e^{iqr}\right) dq$$

$$= -\frac{8\pi}{r}\int_0^\infty \frac{\sin x}{x}\, dx \sim \frac{1}{r}$$

the Fourier transf. of  gives the $\frac{1}{r}$ potential

expect  gives correction to the potential

1-loop



no ghost loops (b^a, c^a don't couple to A_0^a in Coulomb gauge)

- the seagull graph (last one) vanishes in dimension reg.
- propagator up to 1-loop

$$D_{00}^{ab}(q) = \frac{-i\delta^{ab}}{\vec{q}^2}\left[1 - \frac{2ig^2}{\vec{q}^2} C_2(G)(A_{tt} + A_{tc})\right]$$

1st graph : $A_{tt} = \int \frac{d^n k}{(2\pi)^n} (-1)\left(k_0 - \frac{1}{2}q_0\right)^2 \frac{P_{ij}(\vec{k}) P_{ij}(\vec{k} - \vec{q})}{(k^2 - i\varepsilon)[(k-q)^2 - i\varepsilon]}$

2nd graph : $A_{tc} = 2\int \frac{d^n k}{(2\pi)^n} q_i q_j \frac{1}{(\vec{k} - \vec{q})^2} \frac{P_{ij}(\vec{k})}{k^2 - i\varepsilon}$

skipping details

$$A_{tt} = \frac{5}{6}\vec{q}^2 \ln \vec{q}^2 + \text{terms without } \ln \vec{q}^2$$

$$A_{tc} = -\frac{16}{6}\vec{q}^2 \ln \vec{q}^2 + \text{terms without } \ln \vec{q}^2$$

correction to the Coulomb potential ~ Fourier transf of $\frac{1}{\vec{q}^2}(\vec{q}^2 \ln \vec{q}^2)\frac{1}{\vec{q}^2}$

$$\Rightarrow V(r) \sim \frac{g^2}{r}\left(1 + \frac{g^2}{4\pi^2} C_2(G) \frac{11}{6} \ln \frac{r}{r_0}\right)$$

g is defined s.t. at r_0, the Coulomb potential holds

anti screening : 1-loop contribution same sign as tree-level

different from screening in plasma as you move away the effective charge becomes smaller